Synthetic Aperture Imaging

Previous Results:

The radar data collected from a correlation receiver is of the form (weak scatterer model)

$$\eta(\nu,\tau) = \iint_{-\infty}^{\infty} \rho(\nu',\tau')\,\chi(\nu-\nu',\tau-\tau')\,\mathrm{e}^{\mathrm{i}\nu(\tau-\tau')}\mathrm{d}\tau'\mathrm{d}\nu'$$

where ν is (doppler) frequency, τ is time delay, ρ is the scatterer density function, and χ is the ambiguity function associated with the waveform.

For high range resolution waveforms (HRR) we can approximate

$$\chi(\nu-\nu',\tau-\tau') \sim \delta(\tau-\tau')$$

HRR "range profile" (static radar and target)

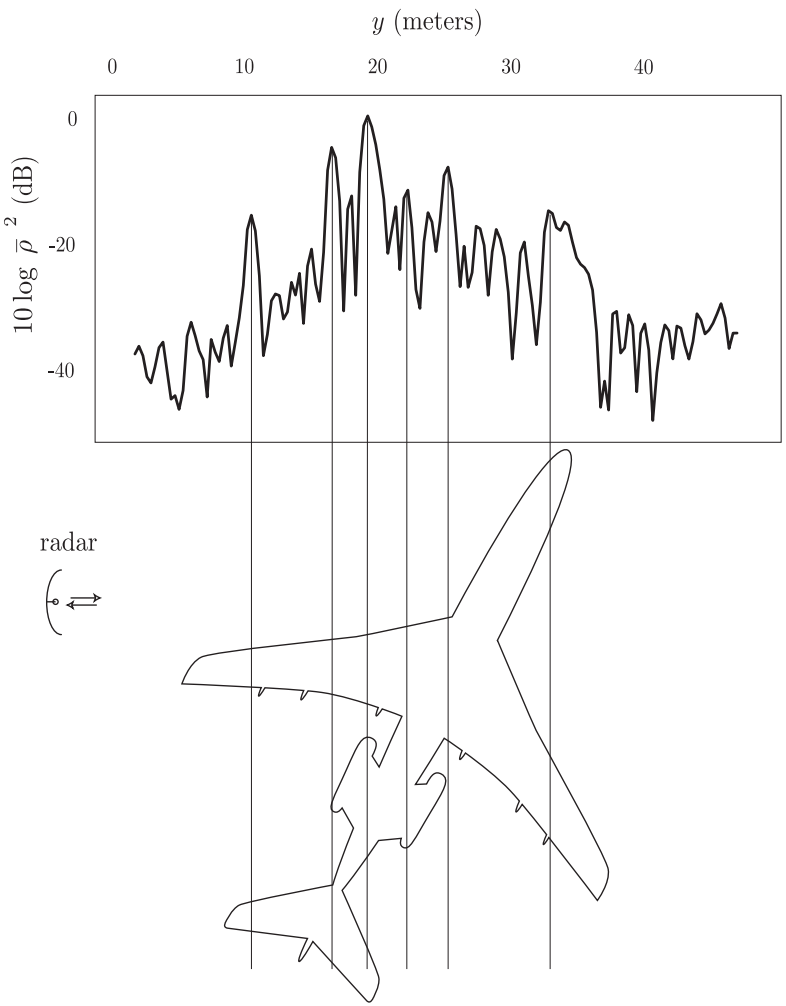


Figure: HRR radar.

Single airborne target

- pulse
  sweeps across target
- can be used
  to form one-dimensional
  "images"
- difficult to
  apply to the problem of
  "target interpretation"

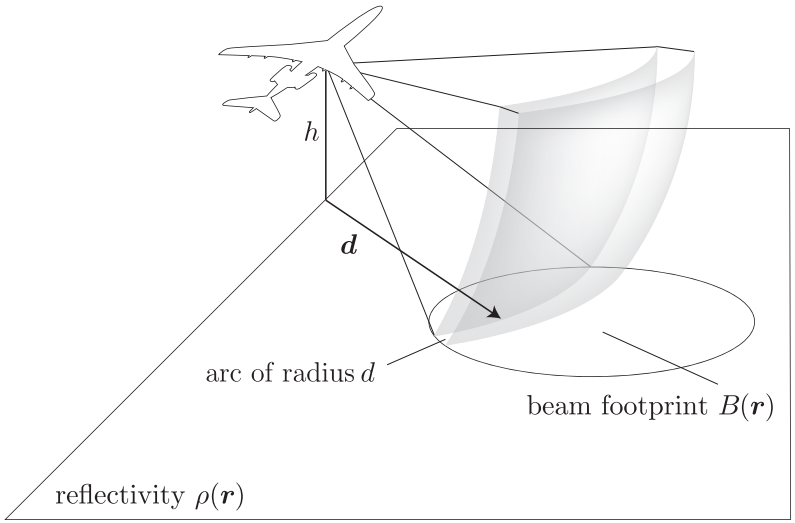


Figure: HRR airborne radar.

Ground illumination

- all scene elements at fixed range illuminated simultaneously
- "ambiguity arcs"

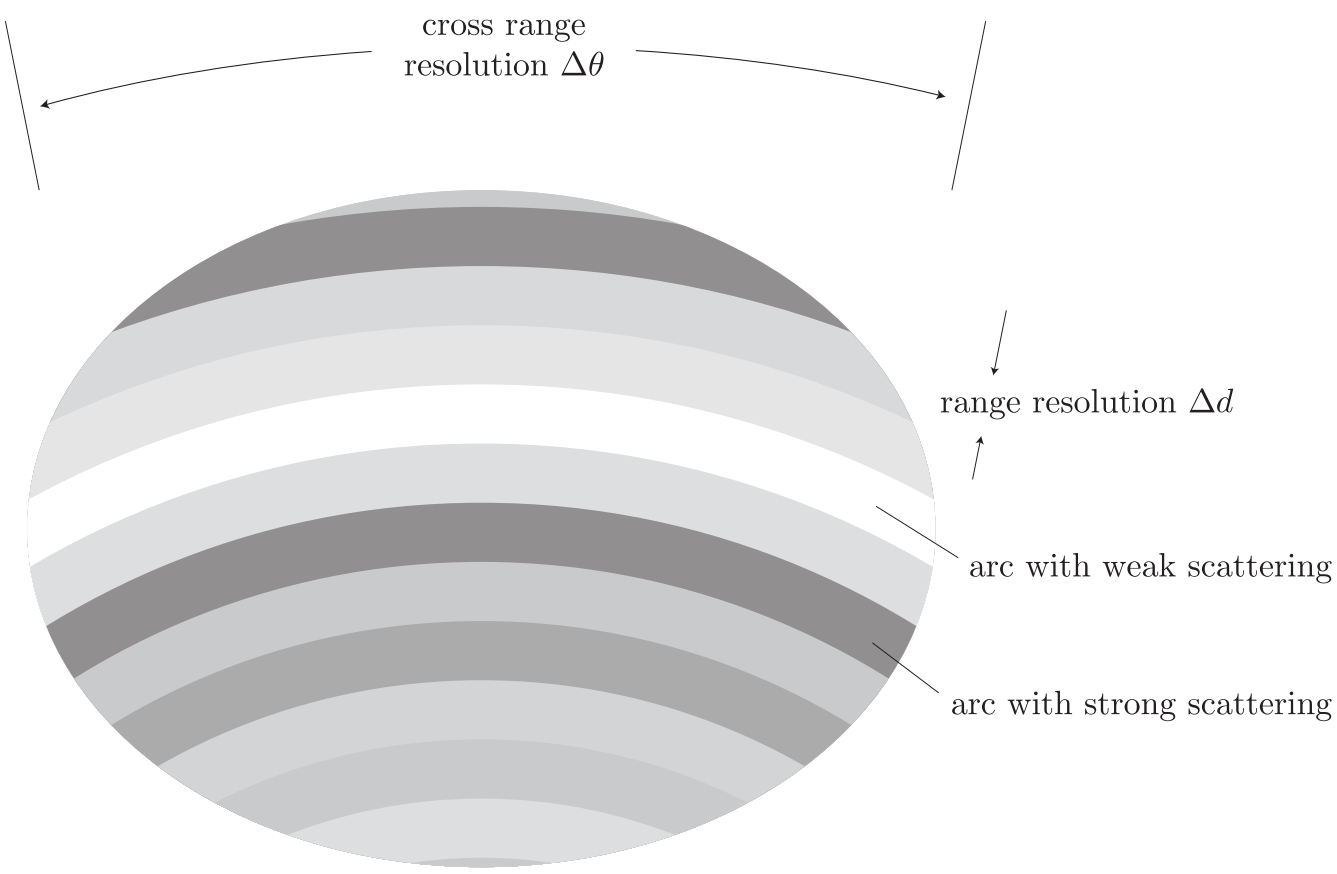


Figure: Ambiguity arcs.

2-D Image Construction

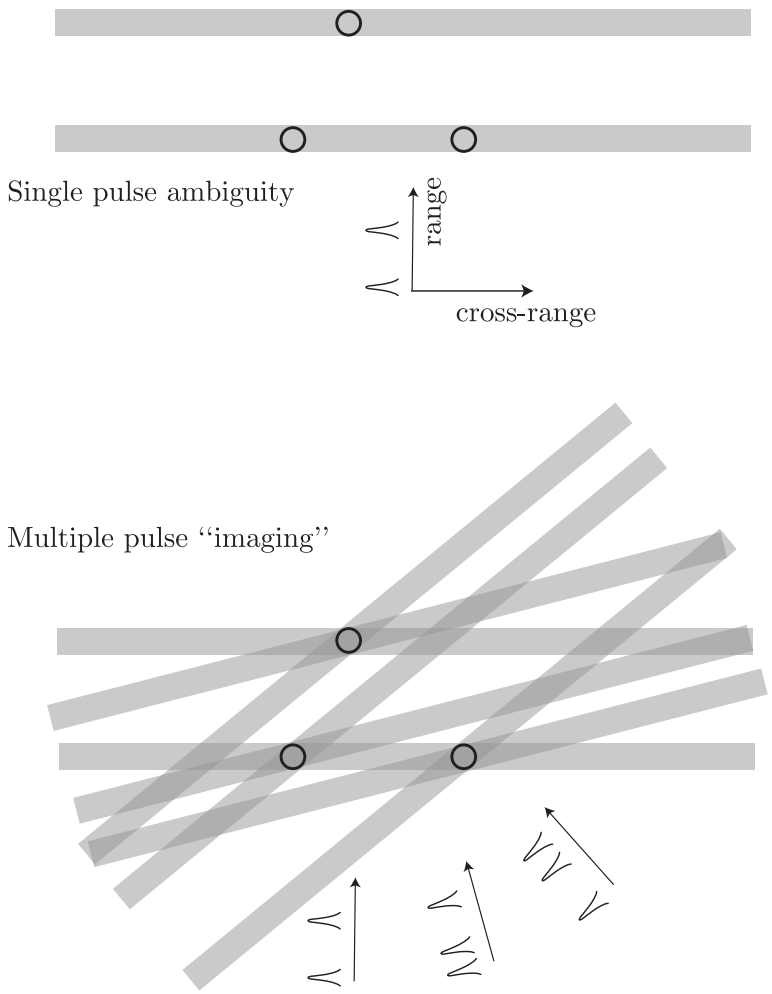


Figure: Back projection.

"triangulation"

- returns
  from multiple directions
- ambiguity
  arcs represent possible
  scatterer locations
- superposition
  disambiguates
  actual locations

Synthetic Apertures

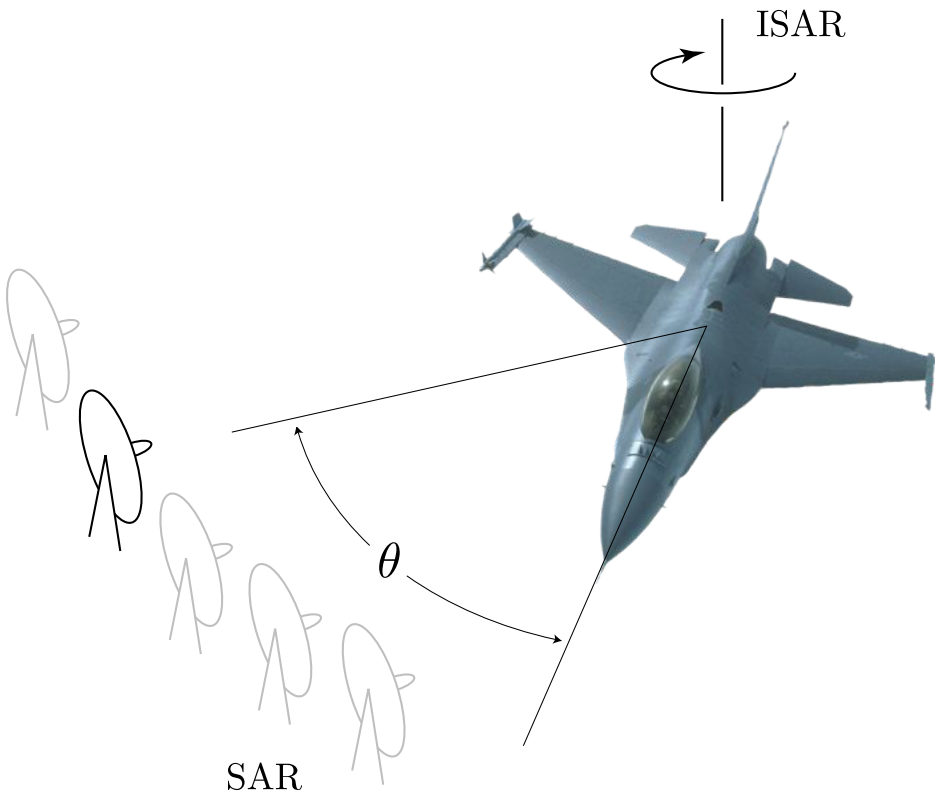


Figure: Two different schemes for forming a synthetic aperture. In SAR systems the radar moves with respect to the target as the data are collected. In ISAR systems the radar is fixed and the target is assumed to rotate.

Rotated HRR Data

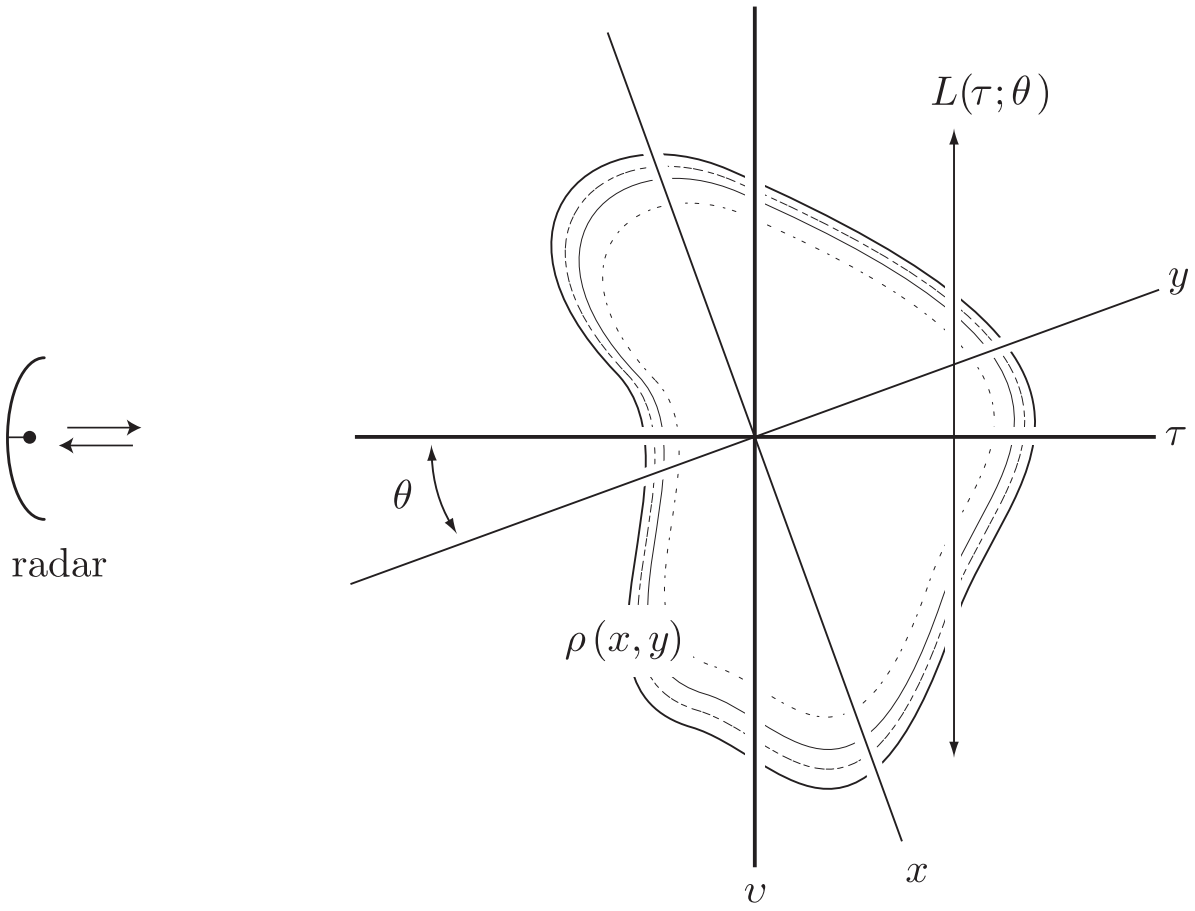


Figure: Geometry of an HRR profile *parameterized by target orientation angle* θ.

Since the scattering density function $\rho_\theta(\nu', \tau')$ will depend on target orientation, HRR data are now of the form:

$$\eta_\theta(\nu, \tau) = \iint_{-\infty}^{\infty} \rho_\theta(\nu', \tau')\, \delta(\tau - \tau') \mathrm{e}^{\mathrm{i}\nu(\tau - \tau')} \mathrm{d}\tau' \mathrm{d}\nu'$$

where η_θ is the range profile of the target at orientation θ.

- $\rho_\theta(\nu', \tau')$ is the rotated version of the one associated with the stationary target
- define a coordinate system x–y fixed to the target and define

$$\rho(x, y) = \rho_{\theta=0}(\nu', \tau')$$

Then for general θ, we have (coordinate rotation)

$$\rho_\theta(\nu', \tau') = \rho(-\tau' \sin\theta + \nu' \cos\theta, \tau' \cos\theta + \nu' \sin\theta)$$

and the data are of the form

$$\eta_\theta(\nu, \tau) = \iint_{-\infty}^{\infty} \rho(-\tau' \sin\theta + \nu' \cos\theta, \tau' \cos\theta + \nu' \sin\theta) \times \delta(\tau - \tau')\, \mathrm{e}^{\mathrm{i}\nu(\tau-\tau')} \mathrm{d}\tau' \mathrm{d}\nu'$$

Change of variables

$$\nu' = x \cos\theta + y \sin\theta \qquad \tau' = -x \sin\theta + y \cos\theta$$

$\Rightarrow$ data are of the form

$$\begin{aligned}\eta_\theta(\nu,\tau) &= \iint_{-\infty}^{\infty} \rho(x,y)\,\delta(\tau + x\sin\theta - y\cos\theta) \\ &\qquad\qquad \times \mathrm{e}^{\mathrm{i}\nu(\tau + x\sin\theta - y\cos\theta)}\mathrm{d}x\mathrm{d}y \\ &= \iint_{-\infty}^{\infty} \rho(x,y)\,\delta(\tau + x\sin\theta - y\cos\theta)\mathrm{d}x\mathrm{d}y\end{aligned}$$

(The complex exponential will have vanishing argument under integration and so drops.)

The Radon Transform

Owing to the δ-function in the integrand, this is just the *line integral* of $\rho(x, y)$ along the line $L(\tau; \theta)$ defined by

$$\tau + x \sin\theta - y \cos\theta = 0$$

- These $\eta_\theta(\nu, \tau)$ are independent of ν
- the double integral restricted to the line $L(\tau; \theta)$ (or the equivalent line integral itself) is known as *the Radon Transform* of $\rho(x, y)$:[1]

$$\overline{\rho}(\tau, \theta) = \{\mathcal{R}\rho\}(\tau, \theta) \equiv \iint_{-\infty}^{\infty} \rho(x, y)\, \delta(\tau + x \sin\theta - y \cos\theta) \mathrm{d}x\mathrm{d}y$$

[1] The Radon Transform was introduced in 1917 by Johann Radon. This transform is important in computerized tomography (CT), where it is known as the "sinogram."

Example: Sinogram of two point scatterers.

$$\rho(x, y) = \delta(x - x_1)\,\delta(y - y_1) + \delta(x - x_2)\,\delta(y - y_2)$$

(two equal strength point scatterers at (x_1, y_1) and (x_2, y_2)).

$$\begin{aligned}\{\mathcal{R}\rho\}(\tau, \theta) &= \iint_{-\infty}^{\infty} [\delta(x - x_1)\,\delta(y - y_1) + \delta(x - x_2)\,\delta(y - y_2)] \\ &\qquad\qquad \times \delta(\tau + x \sin\theta - y \cos\theta)\mathrm{d}x\mathrm{d}y \\ &= \delta(\tau + x_1 \sin\theta - y_1 \cos\theta) + \delta(\tau + x_2 \sin\theta - y_2 \cos\theta) \\ &= \delta(\tau - r_1 \sin(\phi_1 - \theta)) + \delta(\tau - r_2 \sin(\phi_2 - \theta))\end{aligned}$$

where $r_n = \sqrt{x_n^2 + y_n^2}$ and $\phi_n = \arctan(y_n/x_n)$, $n = 1, 2$.
(I.e., a pair of sine curves.)

Projection-Slice Theorem

Fix θ and consider $\overline{\rho}(\tau,\theta) = \overline{\rho}_\theta(\tau)$ to be a function only of τ (that is, one function of τ for each θ). The Fourier transform of $\overline{\rho}_\theta(\tau)$ is

$$
\begin{aligned}
\{\mathcal{F}\,\overline{\rho}_\theta\}(\omega) &= \int_{-\infty}^{\infty} \overline{\rho}_\theta(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau \\
&= \iiint_{-\infty}^{\infty} \rho(x,y)\,\delta(\tau + x\sin\theta - y\cos\theta)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau\mathrm{d}x\mathrm{d}y \\
&= \iint_{-\infty}^{\infty} \rho(x,y)\,\mathrm{e}^{\mathrm{i}\omega(x\sin\theta - y\cos\theta)}\mathrm{d}x\mathrm{d}y \\
&= \iint_{-\infty}^{\infty} \rho(x,y)\,\mathrm{e}^{-\mathrm{i}(\kappa_x x + \kappa_y y)}\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

where $\kappa_x \equiv -\omega\sin\theta$ and $\kappa_y \equiv \omega\cos\theta$.

Note, however, that the Fourier transform of $\rho(x, y)$ in two dimensions is given by

$$P(\kappa_x, \kappa_y) = \iint_{-\infty}^{\infty} \rho(x, y) \mathrm{e}^{-\mathrm{i}(\kappa_x x + \kappa_y y)} \mathrm{d}x\mathrm{d}y$$

so, evidently, we have

$$P(-\omega \sin\theta, \omega \cos\theta) = \{\mathcal{F}\,\overline{\rho}_\theta\}(\omega)$$

This result is known as the "Projection-Slice Theorem" and, since $\rho(x, y)$ can be determined from the inverse Fourier transform of $P(\kappa_x, \kappa_y)$, we can use this theorem to build an inverse Radon transform for $\overline{\rho}(\tau, \theta)$.

Filtered Backprojection

Change of variables $(\kappa_x, \kappa_y) \mapsto (\omega, \theta)$, where $\kappa_x \equiv -\omega \sin\theta$ and $\kappa_y \equiv \omega \cos\theta \Rightarrow$

$$\begin{aligned}
\rho(x,y) &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} P(\kappa_x, \kappa_y) \mathrm{e}^{\mathrm{i}(\kappa_x x + \kappa_y y)} \mathrm{d}\kappa_x \mathrm{d}\kappa_y \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \{\mathcal{F}\,\overline{\rho}_\theta\}(\omega) \mathrm{e}^{-\mathrm{i}\omega(x\sin\theta - y\cos\theta)} \frac{\partial(\kappa_x, \kappa_y)}{\partial(\omega, \theta)} \mathrm{d}\omega \mathrm{d}\theta \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \{\mathcal{F}\,\overline{\rho}_\theta\}(\omega) \mathrm{e}^{-\mathrm{i}\omega(x\sin\theta - y\cos\theta)} |\omega| \mathrm{d}\omega \mathrm{d}\theta \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \Big(|\omega| \{\mathcal{F}\,\overline{\rho}_\theta\}(\omega) \Big) \mathrm{e}^{\mathrm{i}\omega(-x\sin\theta + y\cos\theta)} \mathrm{d}\omega \mathrm{d}\theta
\end{aligned}$$

The inner integral is just the inverse Fourier transform of the product

$$Q(\omega) \equiv |\omega| \times \{\mathcal{F}\,\overline{\rho}_\theta\}(\omega)$$

which behaves like a "smoothing filter" acting on $\overline{\rho}_\theta(\tau)$.

The inverse Fourier transform of $Q(\omega)$ is

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(\omega) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega$$

and so we can write

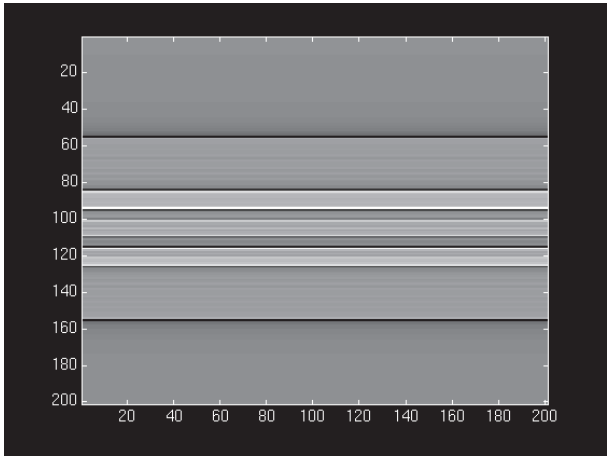
$$\rho(x, y) = \frac{1}{2\pi} \int_{0}^{2\pi} q(-x \sin\theta + y \cos\theta) \mathrm{d}\theta$$

where (substituting for $\{\mathcal{F}\,\overline{\rho}_\theta\}(\omega)$), we obtain

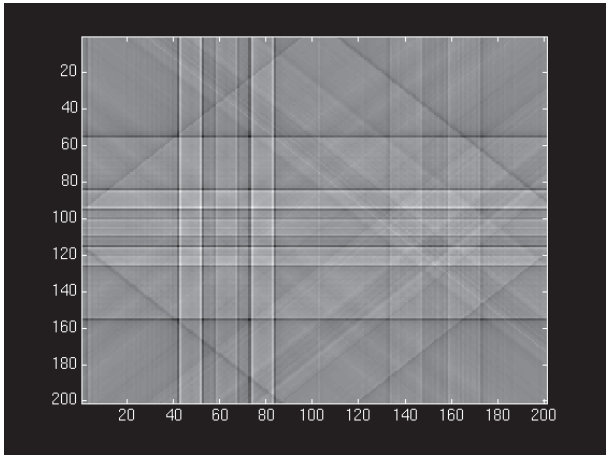
$$q(t) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \overline{\rho}_\theta(\tau)\, |\omega|\, \mathrm{e}^{\mathrm{i}\omega(t-\tau)} \mathrm{d}\omega \mathrm{d}\tau$$

These last two equations are collectively known as the *filtered backprojection algorithm.*

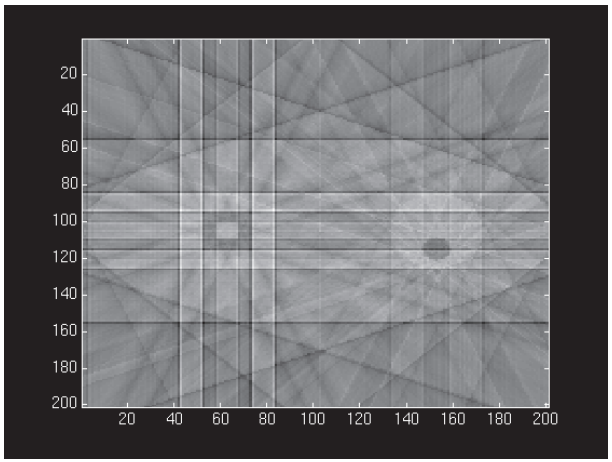
Filtered Backprojection in MatLab: `iradon`



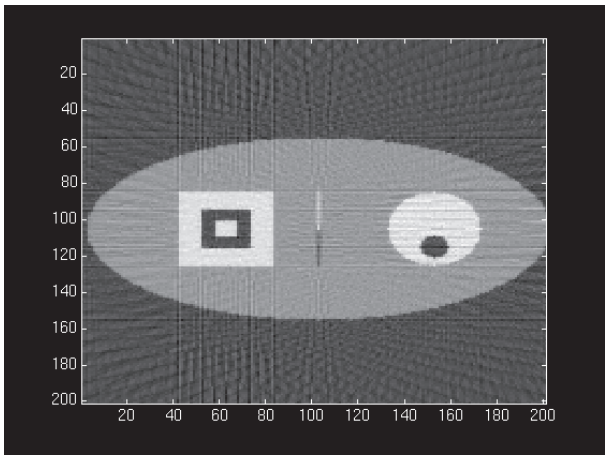
1 view



4 views



8 views



60 views

Figure: Example of filtered backprojection using discrete angle data.

The "Fourier Approach"

When the synthetic aperture $\Delta\theta$ is small so that $0 \leq \theta \leq \Delta\theta \ll 1$, then the small angle approximation yields

$$\begin{aligned}\{\mathcal{F}\,\overline{\rho}_\theta\}(\omega) &= \iint_{-\infty}^{\infty} \rho(x,y)\mathrm{e}^{\mathrm{i}\omega(x\sin\theta - y\cos\theta)}\mathrm{d}x\mathrm{d}y \\ &\approx \iint_{-\infty}^{\infty} \rho(x,y)\mathrm{e}^{\mathrm{i}\omega(x\theta - y)}\mathrm{d}x\mathrm{d}y\end{aligned}$$

Let $k_x \equiv -\omega\theta$ and $k_y \equiv \omega$. For the (frequency domain) data set $\{\mathcal{F}\,\overline{\rho}_\theta\}(\omega) \mapsto P(k_x, k_y)$, we can recover $\rho(x,y)$ by simple Fourier inversion:

$$\rho(x,y) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} P(k_x,k_y)\,\mathrm{e}^{\mathrm{i}(k_x x + k_y y)}\mathrm{d}k_x\mathrm{d}k_y$$

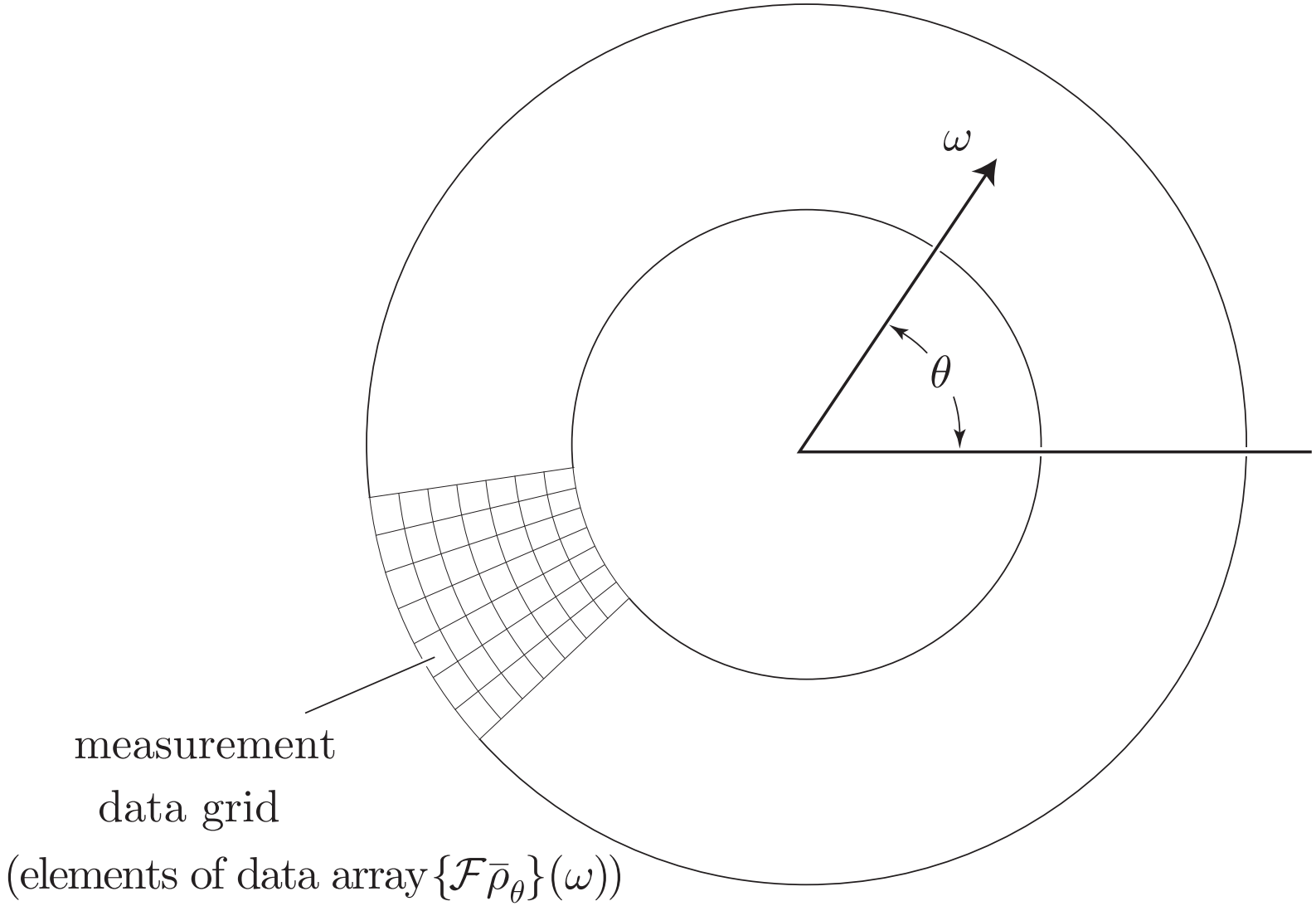


Figure: Data domain. Here, $k_y = \omega$ is a polar radial direction and $k_x = \omega\theta$ is an arclength (for each ω).

Focussing

The data set mapping

$$\{\mathcal{F}\,\overline{\rho}_\theta\}(\omega) \mapsto P(k_x, k_y)$$

is defined over a (small angle) grid *in polar coordinates.*

- Fourier inversion formula is appropriate to data defined on a *rectangular grid*
- When the aperture is sufficiently small and ω is sufficiently large, the polar grid is pretty close to rectangular in shape
- General case: data must be interpolated to a rectangular grid before Fourier inversion
- In the radar community, this additional preprocessing step is known as "focussing"

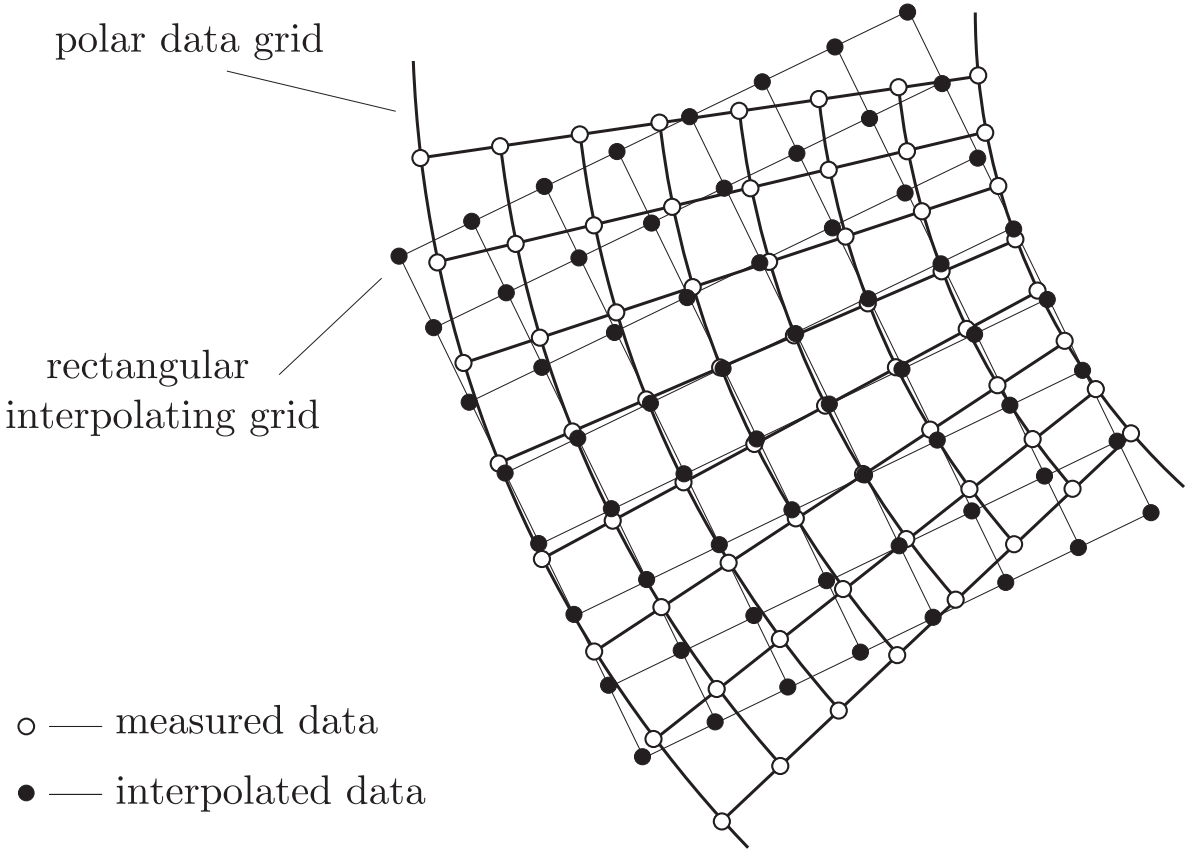


Figure: Data interpolation to a rectangular grid before Fourier inversion.

Resolution

The image is constructed by a Fourier domain version of a convolution with an imaging kernel determined by the inverse Fourier transform of

$$K(k_x, k_y) = \mathrm{e}^{\mathrm{i}(k_x x + k_y y)} \qquad \text{(restricted to finite bandwidth)}$$

For finite bandwidth signals, the spatial-domain version of this imaging kernel is a product of sinc functions and the width of the central lobes provides an estimate of resolution. We can conclude that the dimensions Δx and Δy of a resolution cell are

$$\Delta x \approx \frac{2\pi}{\Delta k_x} \approx \frac{2\pi}{\overline{\omega}\Delta\theta} \quad \text{and} \quad \Delta y \approx \frac{2\pi}{\Delta k_y} \approx \frac{2\pi}{\Delta\omega}$$

where $\overline{\omega}$ is the average frequency, $\Delta\theta$ denotes the aperture size, and $\Delta\omega$ denotes the bandwidth. (This analysis is only appropriate for small-angle apertures.)

Example ISAR Image

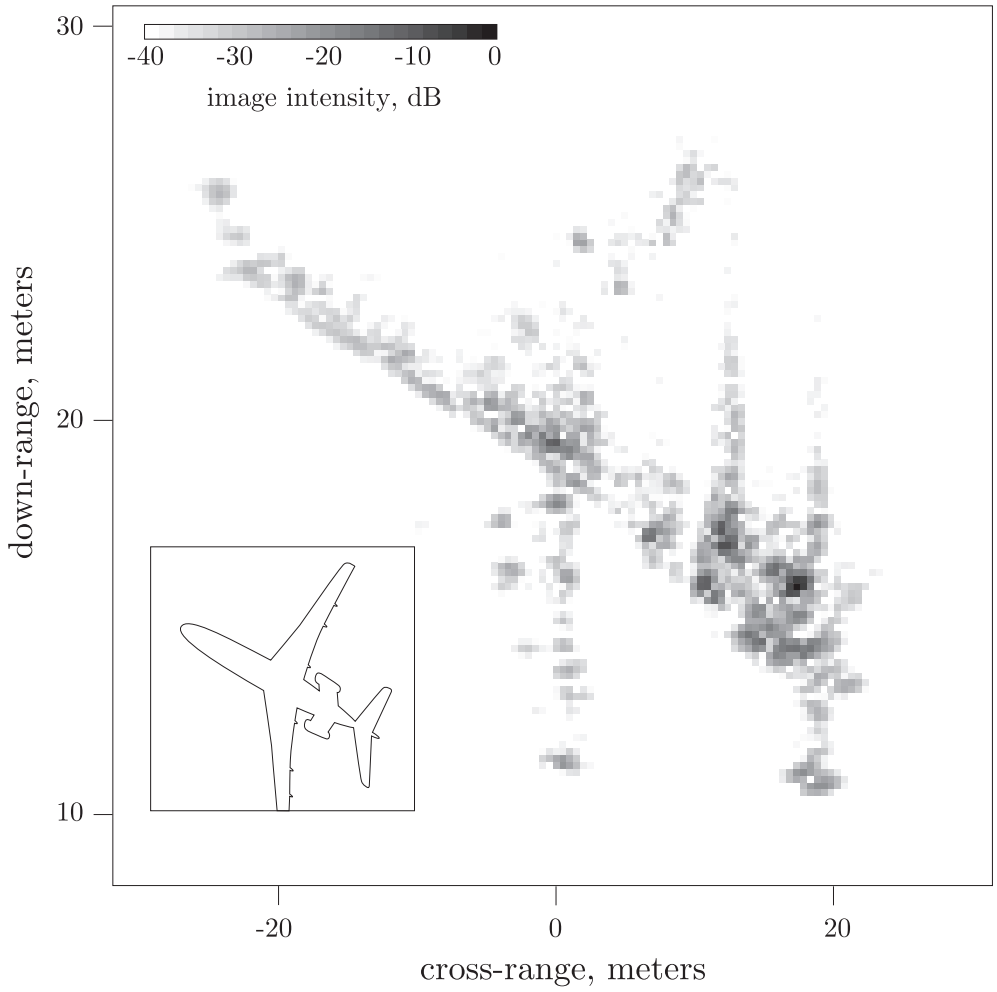


Figure: Example ISAR image of a B-727 jetliner (orientation displayed in inset) created using a radar system with center frequency 9.25 GHz and bandwidth 500 MHz. The synthetic aperture subtended approximately 4°.

Practical Complications

- This imaging method requires the integration of pulses collected over many aspects $\Rightarrow$ the radar must maintain coherence over the synthetic aperture (to avoid destructive interference)
- synthetic aperture data collection process is dynamic — it relies on target/radar relative motion
  - rotation about the body's center of mass
  - translation of that center of mass
- rotation part of the relative radar/target motion used to reconstruct an image
- (usually) while target is rotates *it will also translates*
- target range R contributes to the overall phase of the data (by the factor $\mathrm{e}^{\mathrm{i}kR}$) $\Rightarrow$ unaccounted for variations in R over the aperture will degrade the coherence of the pulse stream and corrupt the final image.

Range Alignment

Can maintain coherence by subtracting target translation effects before imaging

- assume HRR pulse sweeps across the target faster than the target moves ("start-stop" approximation)
- range profile data $\overline{\rho}_\theta(\tau)$ are then of the form $\overline{\rho}_\theta(\tau + \Delta\tau_\theta)$, where $\Delta\tau_\theta$ is a range offset that is determined by target motion *between* pulses.

The collected range profiles can be shifted to a common origin if we can determine $\Delta\tau_\theta$ for each θ.

One method:

- assume that one of the peaks in each of the range profiles (for example, the strongest peak) is always due to the *same* target feature and so provides a convenient origin
- shift all other $\overline{\rho}_\theta$ so that their main peaks align to this origin

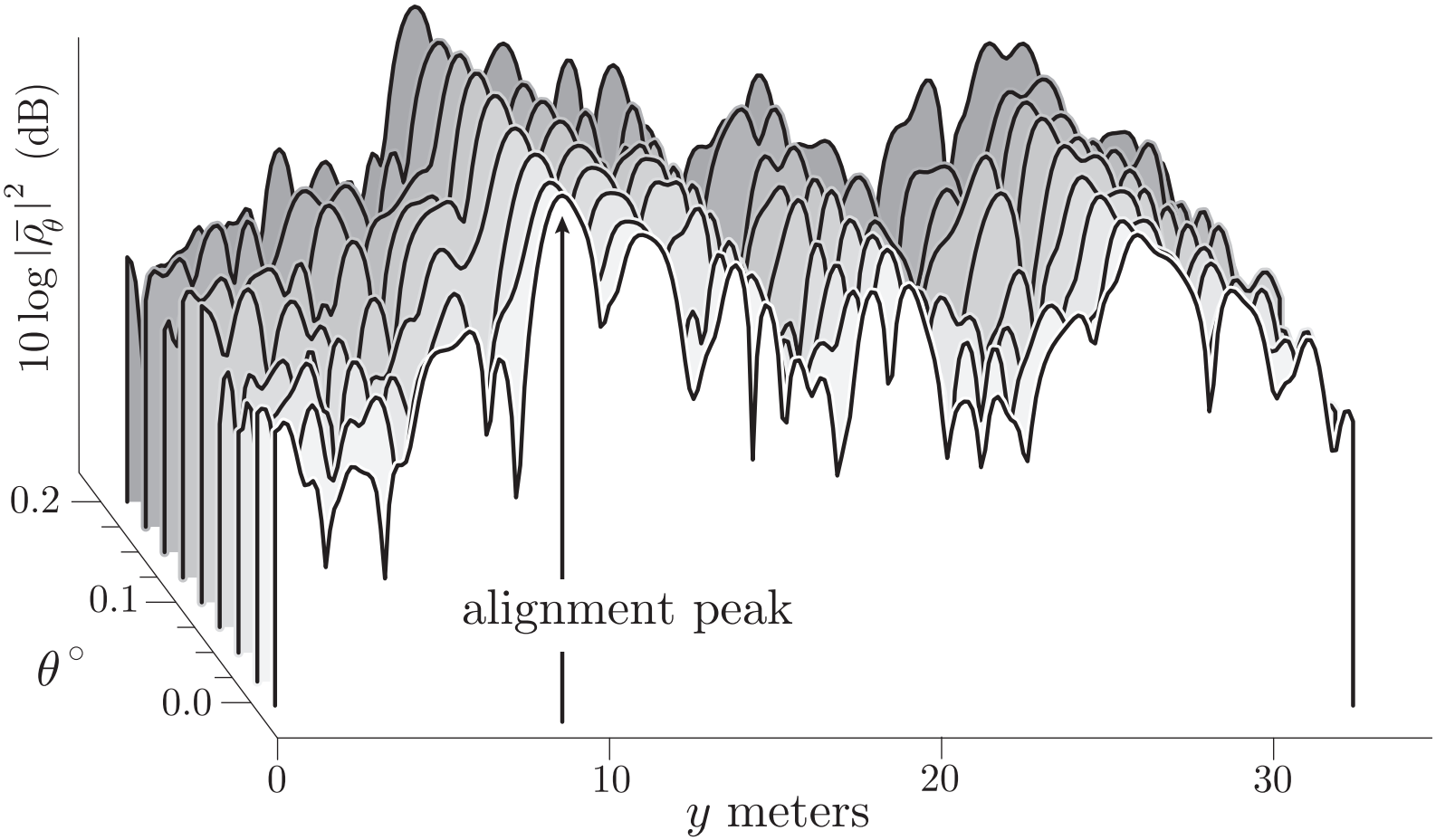


Figure: Range alignment preprocessing in synthetic aperture imaging. The effects of target translation must be removed before backprojection can be applied.

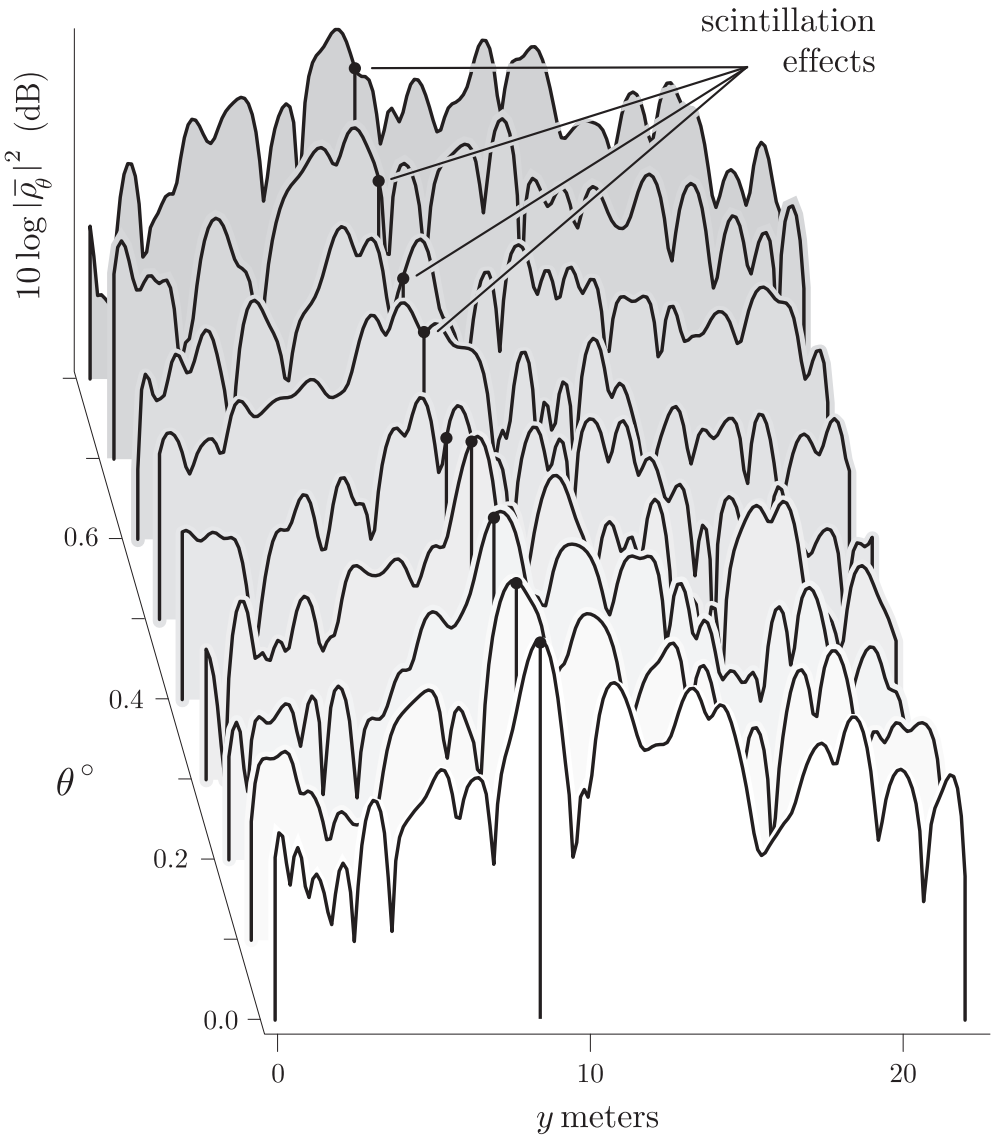


Figure: Alignment peak scintillation ⇒ peak-based alignment may not be possible.

When the strongest peak is not a single point but, rather, several closely spaced and *unresolved* scattering centers, then interference effects can cause the range profile alignment feature to vary rapidly across the synthetic aperture.

For such "scintillating targets," other alignment methods are used: for example, if the target is assumed to move along a "smooth" path, then

$$R(t) = R(0) + \dot{R}t + \tfrac{1}{2}\ddot{R}t^2 + \tfrac{1}{6}\dddot{R}t^3$$

In terms of this polynomial we can write

$$\begin{aligned}\Delta\tau_\theta &= 2\,\frac{R(t_\theta) - R(0)}{c} \\ &= 2\,\frac{\dot{R}t_\theta + \frac{1}{2}\ddot{R}t_\theta^2 + \frac{1}{6}\dddot{R}t_\theta^3}{c}\end{aligned}$$

where t_θ denotes the starting point of the θ-th pulse and $\dot{R}$, $\ddot{R}$, and $\dddot{R}$ are radar measurables.[2]

[2] Typically, this is implemented in the Fourier domain using the shift theorem.